

PRODUCTS OF k -SPACES AND SPACES OF COUNTABLE TIGHTNESS

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ABSTRACT. In this paper, we obtain results of the following type: if $f: X \rightarrow Y$ is a closed map and X is some “nice” space, and Y^2 is a k -space or has countable tightness, then the boundary of the inverse image of each point of Y is “small” in some sense, e.g., Lindelöf or ω_1 -compact. We then apply these results to more special cases. Most of these applications combine the “smallness” of the boundaries of the point-inverses obtained from the earlier results with “nice” properties of the domain to yield “nice” properties on the range.

Introduction. Recall the following theorem due to Morita and Hanai [14] and Stone [17].

THEOREM. *If $f: X \rightarrow Y$ is closed and X is metrizable, then the following are equivalent.*

- (a) Y is first countable;
- (b) For each $y \in Y$, $\partial f^{-1}(y)$ is compact;
- (c) Y is metrizable.

The (c) $\Rightarrow$ (b) part is due to Vainštein [22]. But even the (a) $\Rightarrow$ (b) part holds under much more general conditions: Michael [7] showed (b) holds if X is paracompact, and Y is locally compact or first-countable.

Note that the assumptions on Y in Michael’s theorem could not be weakened to “ Y is a k -space” or “ Y has countable tightness”: the map identifying the limit points of a topological sum of κ convergent sequences is a closed map from a metrizable space X to a Fréchet space Y , and $|\partial f^{-1}(y)| = \kappa$ for some $y \in Y$. In this paper, we show that the situation is different if we require Y^2 to be a k -space or have countable tightness. (Recall that the square of a k -space or a space of countable tightness need not have the same property.) We will usually not be able to show that the boundaries of point-inverses are compact, but we will often (depending upon conditions imposed on X or Y) be able to show that they are “small” in some sense, e.g., Lindelöf or ω_1 -compact. In the second section, we apply general results of this type to more special cases, often combining the “smallness” of the boundaries of point-inverses with “nice” properties of X to obtain “nice” properties of Y .

We mention the following earlier result of the second author [21] which is related to this topic.

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THEOREM. *If $f: X \rightarrow Y$ is closed and X is metrizable, then the following are equivalent.*

- (a) *For each $y \in Y$, $\partial f^{-1}(y)$ is Lindelöf.*
- (b) *Y has a point-countable k -network [see §2, Definition 2.1].*
- (c) *Y has a σ -locally-countable k -network.*

See [7] and [21] for other related results.

We will often make use of the following well-known property of closed maps (cf. [3, p. 52]): If $f: X \rightarrow Y$ is closed, then for each $y \in Y$ and open $U \subset X$ such that $f^{-1}(y) \subset U$, there is a neighborhood V of y such that $f^{-1}(V) \subset U$.

1. General results. All our spaces are assumed to be regular and T_1 . We consider cardinals to be initial ordinals. We now recall some basic definitions.

DEFINITION 1.1. A space X has the *weak topology with respect to a collection $\mathcal{C}$ of sets* if a subset A of X is closed (resp., open) in X if and only if $A \cap C$ is closed (resp., open) in C for each $C \in \mathcal{C}$.

DEFINITION 1.2. A space X is a *k -space (quasi- k -space)* if X has the weak topology with respect to its compact (countably compact) subsets. X is *sequential* if X has the weak topology with respect to its compact metric subspaces (equivalently, with respect to its subspaces homeomorphic to $\omega + 1$, a sequence with its limit point). X has *countable tightness* (denoted by $t(X) \leq \omega$) if it has the weak topology with respect to its countable subsets.

We will be using the following elementary facts about these concepts.

(i) If X has the weak topology with respect to a collection $\mathcal{C}$, and $f: X \rightarrow Y$ is a quotient map, then Y has the weak topology with respect to $\{f(C): C \in \mathcal{C}\}$. Thus all properties named in Definition 1.2 are preserved by quotient maps.

(ii) If X satisfies any of the properties in Definition 2.2 locally, then the whole space has the property.

(iii) If X has a locally finite cover by a family $\mathcal{C}$ of closed sets, then X has the weak topology with respect to $\mathcal{C}$.

DEFINITION 1.3. A space X is (*strongly*) *collectionwise Hausdorff* if whenever $\{x_\alpha: \alpha \in A\}$ is a closed discrete subset of X , there exists a (discrete) disjoint collection $\{U_\alpha: \alpha \in A\}$ of open sets such that $x_\alpha \in U_\alpha$ for each $\alpha \in A$.

Note that every normal collectionwise Hausdorff space is strongly collectionwise Hausdorff.

Let c denote the cardinality of the continuum.

THEOREM 1.4. *Suppose $f: X \rightarrow Y$ is closed, with X strongly collectionwise Hausdorff. Then the boundary, $\partial f^{-1}(y)$, of $f^{-1}(y)$ is c -compact for each $y \in Y$ if either*

- (a) *Y^2 is quasi- k and $t(Y) \leq \omega$ or*
- (b) *$t(Y^2) \leq \omega$.*

PROOF. Suppose $\partial f^{-1}(y)$ is not c -compact. Then there is a closed discrete subset $D \subset \partial f^{-1}(y)$, with $|D| = c$. For each $d \in D$, let U'_d be an open set containing d such that $\{U'_d: d \in D\}$ is discrete.

Let $d \in U_d \subset \bar{U}_d \subset U'_d$, where U_d is open. Note that $\{U_d: d \in D\}$ is also discrete.

For each $d \in D$, $y \in \overline{f(U_d - f^{-1}(y))}$. Since $t(Y) \leq \omega$, there is a countable set $\{y_{d,n} : n \in \omega\} \subset f(U_d - f^{-1}(y))$ such that $y \in \overline{\{y_{d,n} : n \in \omega\}}$.

Let $X_{d,n} = f^{-1}(y_{d,n}) \cap \bar{U}_d$, and let $X_d = f^{-1}(y) \cap \bar{U}_d$.

If O is open and contains X_d , then there is an open set O' such that $O' \cap \bar{U}_d = \emptyset$, and $f^{-1}(y) \subset O \cup O'$. Let W be the complement in Y of $f(X - (O \cup O'))$. Then $y \in W$, so there is $n \in \omega$ such that $y_{d,n} \in W$, and hence $f^{-1}(y_{d,n}) \subset O \cup O'$. Thus $X_{d,n} \subset (O \cup O') \cap \bar{U}_d = O \cap \bar{U}_d \subset O$. Choose $x_{d,n} \in X_{d,n}$, and let $A_d = \{x_{d,n} : n \in \omega\}$. By the above argument, every open set containing X_d contains infinitely many elements of A_d .

For $x \in A_d$, let $D_x = \{d' \in D : \text{there exists } x' \in \bar{U}_{d'} \text{ with } f(x) = f(x')\}$. Let $B_d = \{x \in A_d : D_x \text{ is uncountable}\}$.

Claim 1. $X_d \cap \bar{B}_d = \emptyset$. To see this, let $B_d = \{x_0, x_1, \dots\}$. Inductively choose a sequence $d_0, d_1, \dots$ of distinct elements of D , and points $x'_n \in \bar{U}_{d_n}$ such that $f(x'_n) = f(x_n)$. Then $\{x'_0, x'_1, \dots\}$ is a closed subset of X , so $f(\{x'_0, x'_1, \dots\}) = f(B_d)$ is closed. Thus $y \notin f(B_d) = \overline{f(B_d)} = f(\bar{B}_d)$, so $X_d \cap \bar{B}_d = \emptyset$.

Let $C_d = A_d - B_d$. By Claim 1, $X_d \cap \bar{C}_d \neq \emptyset$. Pick $d(0) \in D$. Let $D(d(0)) = \bigcup \{D_x : x \in \bar{C}_{d(0)}\}$. Observe that $D(d(0))$ is countable. If $d(\beta)$ has been chosen for all $\beta < \alpha < c$, let $d(\alpha) \in D - \bigcup_{\beta < \alpha} D(d(\beta))$. Observe that if $x \in C_{d(\beta)}$ and $x' \in C_{d(\alpha)}$ with $\alpha \neq \beta$, then $f(x) \neq f(x')$.

Now let $\mathcal{E} = \{E_\alpha : \alpha < c\}$ index all subsets of $\bigcup_{n \in \omega} C_{d(n)}$ such that $|E_\alpha \cap C_{d(n)}| = 1$ for each $\alpha < c$ and $n \in \omega$. Let $E_\alpha = \{e_{\alpha n} : n \in \omega\}$ such that $e_{\alpha n} \in C_{d(n)}$. Let $C_{d(\alpha)} = \{c_{\alpha n} : n \in \omega\}$.

For $x \in X$, denote $f(x)$ by x^* . Let $H_\alpha = \{(e_{\alpha n}^*, c_{\lambda_{\alpha n}}^*) : n \in \omega\} \subset Y^2$, where λ_α is the α th limit ordinal, and let $H = \bigcup_{\alpha < c} H_\alpha$.

Claim 2. $(y, y) \in \bar{H} - H$. To see this, suppose $y \in O$, O open in Y . We know $X_{d(\alpha)} \cap \bar{C}_{d(\alpha)} \neq \emptyset$, so $y \in f(\bar{C}_{d(\alpha)}) = \overline{f(C_{d(\alpha)})}$. For each $n \in \omega$, choose $x_n \in C_{d(n)}$ such that $x_n^* \in O$. Then $\{x_n : n \in \omega\} = E_\alpha$ for some α , and $e_{\alpha n} = x_n$ for each n . There is $n \in \omega$ such that $c_{\lambda_{\alpha n}}^* \in O$. Thus $(e_{\alpha n}^*, c_{\lambda_{\alpha n}}^*) \in O^2 \cap H_\alpha$, which proves the claim.

The next claim completes the proof of part (a).

Claim 3. If $K \subset Y^2$ is countably compact, then $K \cap H$ is finite. To see this, suppose $\alpha_0, \alpha_1, \dots$ are distinct ordinals such that for each $n \in \omega$, $K \cap H_{\alpha_n} \neq \emptyset$. Then we can find $(e_{\alpha_n k_n}^*, c_{\lambda_{\alpha_n k_n}}^*) \in K \cap H_{\alpha_n}$. But $\{c_{\lambda_{\alpha_n k_n}}^* : n \in \omega\}$ is a closed discrete subset of X , since $c_{\lambda_{\alpha_n k_n}} \in \bar{U}_{d(\lambda_{\alpha_n k_n})}$. Thus $\{(e_{\alpha_n k_n}^*, c_{\lambda_{\alpha_n k_n}}^*) : n \in \omega\}$ is an infinite closed discrete subset of K , contradiction. Thus K meets only finitely many H_α 's. Now suppose that for fixed α , $K \cap H_\alpha$ is infinite. Then for each $n \in \omega$, we can find $(e_{\alpha k_n}^*, c_{\lambda_{\alpha k_n}}^*) \in K \cap H_\alpha$. But $\{e_{\alpha k_n} : n \in \omega\}$ is an infinite closed discrete subset of X and we get a contradiction as before. Thus each $K \cap H_\alpha$ is finite, and so $K \cap H$ is finite.

To complete the proof of part (b), we have the next claim.

Claim 4. No countable subset of H contains (y, y) in its closure. Suppose $C \subset H$, $|C| \leq \omega$. Then there exists a sequence $\alpha_0, \alpha_1, \dots$ of distinct ordinals such that $C \subset \bigcup_{n \in \omega} H_{\alpha_n}$. For each n , let $U_n \subset U'_d$ be an open set in X containing $X_{d(n)}$ such that $e_{\alpha_n k} \notin U_n$ if $k \leq n$. Note $U_n \cap \bar{U}_d = \emptyset$ if $d \neq d(n)$. Let V_n be an open set in X

containing $X_{d(\lambda_{\alpha_n})}$ such that $c_{\lambda_{\alpha_n k}} \notin V_n$ if $k \leq n$, and $V_n \cap \bar{U}_d = \emptyset$ if $d \neq d(\lambda_{\alpha_n})$.

If $x \in f^{-1}(y) - \bigcup_{n \in \omega} (U_n \cup V_n)$, then there is an open set O_x containing x such that $O_x \cap \bar{U}_d = \emptyset$ if $d \in \{d(n); n \in \omega\} \cup \{d(\lambda_{\alpha_n}); n \in \omega\}$. Let

$$O = \bigcup \left\{ O_x : x \in f^{-1}(y) - \bigcup_{n \in \omega} (U_n \cup V_n) \right\} \cup \left(\bigcup_{n \in \omega} (U_n \cup V_n) \right)$$

There is an open set W containing y such that $f^{-1}(W) \subset O$. Suppose $W^2 \cap C \neq \emptyset$. Then there exists m such that $W^2 \cap H_{\alpha_m} \neq \emptyset$. Choose $n \in \omega$ such that $(e_{\alpha_m n}^*, c_{\lambda_{\alpha_m n}}^*) \in W^2 \cap H_{\alpha_m}$. Then $(e_{\alpha_m n}, c_{\lambda_{\alpha_m n}}) \in O^2$. Recall $e_{\alpha_m} \in C_{d(n)} \subset \bar{U}_{d(n)}$. Thus $e_{\alpha_m n} \in U_n$, and so $m > n$. Also recall $c_{\lambda_{\alpha_m n}} \in C_{d(\lambda_{\alpha_m})} \subset \bar{U}_{d(\lambda_{\alpha_m})}$. Thus $c_{\lambda_{\alpha_m n}} \in V_m$, which means $n > m$, a contradiction. This proves Claim 4. Hence Y^2 does not have countable tightness, a contradiction which proves the theorem. $\square$

Assuming the continuum hypothesis (CH), we have the following corollary.

COROLLARY 1.5 (CH). *Suppose $f: X \rightarrow Y$ is closed, with X paracompact. Then each $\partial f^{-1}(y)$ is Lindelöf if either Y^2 is a k -space with $t(Y) \leq \omega$, or $t(Y^2) \leq \omega$.*

PROOF. Immediate from Theorem 1.4 and the fact that ω_1 -compact paracompact spaces are Lindelöf [1].

REMARK. By the proof below, if Y^2 is a k -space with $t(Y) \leq \omega$, then $t(Y^2) \leq \omega$. Thus the two conditions are not independent.

PROOF. Since Y^2 is a k -space, it has the weak topology with respect to the collection of compact subsets of Y^2 ; that is $A \subset Y^2$ is closed whenever $A \subset C$ is closed in C for every compact subset C of Y^2 . Each compact subset C of Y^2 is contained in $\pi(C)^2$, where π is the projection from Y^2 onto Y . Then Y^2 has the weak topology with respect to $\{\pi(C)^2; C \text{ is compact in } Y^2\}$. Since each $\pi(C)$ is a compact space of countable tightness, by a result of V. I. Malyhin [5, Theorem 4], so is each $\pi(C)^2$. Then $t(Y^2) \leq \omega$. $\square$

We do not know if Corollary 1.5 is true without CH. The problem seems to hinge on strengthening the conclusion of Theorems 1.3 and 1.4 by replacing “ c -compact” with “ ω_1 -compact”. It turns out if we add the condition “ Y is sequential” to the hypotheses of these theorems, then we can do it.

THEOREM 1.6. *Suppose $f: X \rightarrow Y$ is closed with X strongly collectionwise Hausdorff and Y sequential. Then each $\partial f^{-1}(y)$ is ω_1 -compact if either Y^2 is a quasi- k -space or $t(Y^2) \leq \omega$.*

PROOF. Suppose Y^2 is a quasi- k -space. Since Y is sequential, by [18, Theorem 2.2] Y^2 is sequential, hence $t(Y^2) \leq \omega$. Thus we can assume that $t(Y^2) \leq \omega$. Suppose $\partial f^{-1}(y)$ is not ω_1 -compact. Then there is a closed discrete set $D \subset \partial f^{-1}(y)$ with $|D| = \omega_1$. Let $\{U_d; d \in D\}$ be a discrete collection of open sets in X with $d \in U_d$. Then $f(\bar{U}_d)$ is a closed subset of the sequential space Y , and is therefore sequential. Since y is not isolated in $f(\bar{U}_d)$, there exists a sequence $y_{d,n} \rightarrow y$, with $y_{d,n} \in f(\bar{U}_d) - \{y\}$ for each $n \in \omega$. Choose $x_{d,n} \in \bar{U}_d \cap f^{-1}(y_{d,n})$. As in the proof of Theorem 1.3, we can construct $\{d(\alpha); \alpha < \omega_1\} \subset D$ and an infinite set $C_{d(\alpha)} \subset \{x_{d(\alpha),n}; n \in \omega\}$ such that f is 1-1 on $\bigcup_{\alpha < \omega_1} C_{d(\alpha)}$. Let $X_{d(\alpha)} = \bar{U}_{d(\alpha)} \cap f^{-1}(y)$. Observe that every open set containing $X_{d(\alpha)}$ contains all but finitely many elements of $C_{d(\alpha)}$, and that

$f^{-1}(y) \cup (\bigcup_{\alpha < \omega_1} C_{d(\alpha)})$ is closed. Thus Y contains a closed copy of the space obtained by identifying the limit points of ω_1 convergent sequences. In [4], this space is denoted by S_{ω_1} , and it is proved there that $S_{\omega_1}^2$ is not a k -space.

To complete the proof, it is sufficient to show that $S_{\omega_1}^2$ does not have countable tightness. For each $\alpha < \omega_1$, let $S_\alpha \subset S_{\omega_1}$ be the union of the first α sequences (with limit point). The closure of a countable subset of $S_{\omega_1}^2$ is contained in some S_α^2 . Thus if $S_{\omega_1}^2$ had countable tightness, then it would have the weak topology with respect to $\{S_\alpha^2; \alpha < \omega_1\}$. But each S_α^2 is a k -space (cf. [9, (7.5)]), so then $S_{\omega_1}^2$ would be a k -space, contradiction. $\square$

COROLLARY 1.7. *Suppose $f: X \rightarrow Y$ is closed with X paracompact and Y sequential. Then each $\partial f^{-1}(y)$ is Lindelöf if either Y^2 is a k -space or $t(Y^2) \leq \omega$.*

The following example shows that the assumption “ Y^2 is a k -space” is not sufficient to obtain “ $\partial f^{-1}(y)$ Lindelöf” in Corollary 1.5.

EXAMPLE 1.8. There exists $f: X \rightarrow Y$ closed with X locally compact and paracompact, such that Y^2 is a k -space, but $\partial f^{-1}(y)$ is not Lindelöf for some $y \in Y$.

PROOF. For each $\alpha < \omega_1$, let $S(\alpha)$ be a copy of ordinal space $\omega_1 + 1$. Let X be the free union of $\{S(\alpha); \alpha < \omega_1\}$. Let Y be the space obtained from X by identifying the point ω_1 in each copy to a single point ∞ . Let $f: X \rightarrow Y$ be the quotient map. Then X is paracompact and locally compact, f is closed, and $\partial f^{-1}(\infty)$ is not Lindelöf. X is a k -space (being locally compact), hence so is Y .

It remains to prove that Y^2 is a k -space. First we introduce some notation. For each $\alpha, \beta \leq \omega_1$, let $\beta(\alpha)$ be the image under f of the element of $S(\alpha)$ corresponding to the ordinal number β . If $\beta < \beta' \leq \omega_1$, let $[\beta(\alpha), \beta'(\alpha)] = \{\gamma(\alpha); \beta \leq \gamma \leq \beta'\}$, and let $[\beta(\alpha), \infty] = [\beta(\alpha), \omega_1(\alpha)]$.

Suppose $A \subset Y^2$, with A k -closed, but not closed. Since for each $\alpha, \beta < \omega_1$, $[O(\alpha), \beta(\alpha)] \times Y$ and $Y \times [O(\alpha), \beta(\alpha)]$ are clopen k -subspaces of Y^2 , it must be true that $(\infty, \infty) \in \bar{A} - A$.

Since $[f(S(O))]^2 \cap A$ is closed, there exists $\gamma_0 < \omega_1$ such that $[\gamma_0(O), \infty]^2 \cap A = \emptyset$. Now suppose γ_α has been defined for all $\alpha < \beta$, where $\beta < \omega_1$, in such a way that the following property P_α holds.

$$P_\alpha: (\beta_1(\alpha_1), \beta_2(\alpha_2)) \in A \quad \text{and} \quad \alpha_1, \alpha_2 \leq \alpha \text{ implies } \beta_1 < \gamma_{\alpha_1} \text{ or } \beta_2 < \gamma_{\alpha_2}.$$

It is easy to check that P_0 holds from the way γ_0 has been defined. We will show how to define γ_β in such a way that P_β holds.

For each $\alpha \leq \beta$, $f(S(\alpha) \times S(\beta)) \cap A$ and $f(S(\beta) \times S(\alpha)) \cap A$ are closed, so there exists $\delta_{\alpha, \beta} < \omega_1$ such that

- (i) $\gamma_\alpha \leq \delta_{\alpha, \beta}$;
- (ii) $([\delta_{\alpha, \beta}(\alpha), \infty] \times [\delta_{\alpha, \beta}(\beta), \infty]) \cap A = \emptyset$; and
- (iii) $([\delta_{\alpha, \beta}(\beta), \infty] \times [\delta_{\alpha, \beta}(\alpha), \infty]) \cap A = \emptyset$.

For each $\alpha < \beta$, and each $\beta' \in [\gamma_\alpha, \delta_{\alpha, \beta})$, we have by P_α that $(\beta'(\alpha), \infty) \notin A$ and $(\infty, \beta'(\alpha)) \notin A$. Thus there exists $\delta_{\beta', \beta}^\alpha < \omega_1$ such that

- (a) $([\delta_{\beta', \beta}^\alpha(\beta), \infty] \times \{\beta'(\alpha)\}) \cap A = \emptyset$ and
- (b) $(\{\beta'(\alpha)\} \times [\delta_{\beta', \beta}^\alpha(\beta), \infty]) \cap A = \emptyset$.

Let $\gamma_\beta = \sup\{\delta_{\alpha,\beta} : \alpha \leq \beta\} + \sup\{\delta_{\beta',\beta}^\alpha : \alpha < \beta, \gamma_\alpha \leq \beta' < \delta_{\alpha,\beta}\}$. Then $\gamma_\beta < \omega_1$. To show that P_β holds, we can suppose $(\beta_1(\alpha), \beta_2(\beta)) \in A$, with $\alpha \leq \beta$. If $\alpha = \beta$, then either $\beta_1 < \delta_{\beta,\beta} \leq \gamma_\beta$ or $\beta_2 < \delta_{\beta,\beta} \leq \gamma_\beta$, so P_β holds. If $\alpha < \beta$, we can suppose $\beta_2 \geq \gamma_\beta$. Then it must be true that $\beta_1 < \delta_{\alpha,\beta}$ (by (ii) above). If $\gamma_\alpha \leq \beta_1 < \delta_{\alpha,\beta}$, then since $\gamma_\beta \geq \delta_{\beta_1,\beta}^\alpha$, we have $(\beta_1(\alpha), \beta_2(\beta)) \in (\{\beta_1(\alpha)\} \times [\delta_{\beta_1,\beta}^\alpha(\beta), \infty]) \cap A$, a contradiction. Thus $\beta_1 < \gamma_\alpha$, so P_β holds.

Thus we can define $\{\gamma_\alpha : \alpha < \omega_1\}$ in such a way that P_α holds for each $\alpha < \omega_1$. Let $U = \{\beta(\alpha) : \beta > \gamma_\alpha, \alpha < \omega_1\}$. Then U is an open set in Y containing ∞ . Since $(\infty, \infty) \in \bar{A}$, there exists $(\beta_1(\alpha_1), \beta_2(\alpha_2)) \in U^2 \cap A$. Since $P_{\alpha_1+\alpha_2}$ holds, either $\beta_1 < \gamma_{\alpha_1}$ or $\beta_2 < \gamma_{\alpha_2}$. But then either $\beta_1(\alpha_1) \notin U$ or $\beta_2(\alpha_2) \notin U$, contradiction. Thus Y^2 is a k -space.

2. Applications. As applications of results in §1, we shall consider the products of k -spaces and spaces of countable tightness in more special cases.

DEFINITION 2.1 [8, 16]. A collection $\mathcal{P}$ of (not necessarily open) subsets of a space X is a k -network for X if, whenever $C \subset U$ with C compact and U open, then $C \subset \bigcup \mathcal{F} \subset U$ for some finite subcollection $\mathcal{F}$ of $\mathcal{P}$. An $\aleph$ -space is a space with a σ -locally finite k -network, and an $\aleph_0$ -space is a space with a countable k -network.

Note that metrizable spaces are $\aleph$ -spaces, and separable metrizable spaces are $\aleph_0$ -spaces.

We say that X is a *locally $\aleph_0$ -space* if each point of X has a neighborhood which is an $\aleph_0$ -space.

THEOREM 2.2 (CH) *Let $f: X \rightarrow Y$ be a closed map. Let X be a paracompact, locally $\aleph_0$ -space. Then the following are equivalent.*

- (a) $t(Y^2) \leq \omega$;
- (b) each $\partial f^{-1}(y)$ is Lindelöf;
- (c) Y is a locally $\aleph_0$ -space; and
- (d) Y is locally separable.

Furthermore, if Y is sequential, then the CH assumption can be omitted.

PROOF. (a) $\Rightarrow$ (b): This is Corollary 1.5.

(b) $\Rightarrow$ (c): Since each subset of a locally $\aleph_0$ -space is locally $\aleph_0$, as in the proof of [7, Corollary 1.2], we can assume that each $f^{-1}(y)$ is Lindelöf. Thus, f is a closed map with each $f^{-1}(y)$ Lindelöf. Then, for each $y \in Y$, there is a closed neighborhood W of y in Y , and open subsets V_i of X which are $\aleph_0$ -spaces such that $f^{-1}(W) \subset \bigcup_{i=1}^\infty V_i$. Since $\bigcup_{i=1}^\infty V_i$ is an $\aleph_0$ -space, so is $f^{-1}(W)$. Since the closed image of an $\aleph_0$ -space is also $\aleph_0$ by [8, G], W is an $\aleph_0$ -space. This implies (c).

(c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (d): By [8, F], Y^2 is a locally $\aleph_0$ -space. Then, by [8, D, E] Y^2 is locally a hereditarily separable space. Hence $t(Y^2) \leq \omega$.

(d) $\Rightarrow$ (b): This follows from [21, Proposition 1], because Y is paracompact, hence is locally Lindelöf by (d).

From Theorem 2.2 and some results in [21], we have

COROLLARY 2.3. *Let $f: X \rightarrow Y$ be a closed map with X locally separable metric. Then the following are equivalent.*

- (a) $i(Y^2) \leq \omega$;
- (b) each $\partial f^{-1}(y)$ is Lindelöf;
- (c) Y is locally separable;
- (d) Y is locally Lindelöf;
- (e) Y is an $\aleph$ -space.

DEFINITION 2.4. A decreasing sequence (A_n) in a space X is a k -sequence [10], if it is an outer network at a compact subset K of X ; that is, $K = \bigcap_{n=1}^{\infty} A_n$ and every neighborhood of K contains some A_n . By regularity, if a compact subset K has an outer network, it has one in which each A_n is closed in X .

Let Y be a space. Then Y satisfies condition $K(\aleph_0)$ [20] if, for any k -sequence (A_n) in Y , some A_n is countably compact.

It is shown that [20, Proposition 2.4] a space Y satisfies $K(\aleph_0)$ if and only if each closed subset of Y which is a paracompact M -space is locally compact.

DEFINITION 2.5 [10, Lemma 3.E.2]. A space Y is a bi - k -space if, whenever a filter base $\mathcal{F}$ accumulates at y in Y , then there exists a k -sequence (A_n) in Y such that $y \in \overline{F \cap A_n}$ for all $n \in N$ and all $F \in \mathcal{F}$.

It is shown that [10, Theorem 3.E.3] Y is a bi - k -space if and only if Y is a bi-quotient image of a paracompact M -space X . Then by a result of H. Wicke [23], spaces of pointwise countable type [2] are bi - k .

DEFINITION 2.6. A space X is a k_ω -space [9] (K. Morita [13] called it a *space of class* $\mathfrak{S}'$), if it has the weak topology with respect to a countable covering of compact subsets of X .

For a space Y we shall say that Y is a *locally k_ω -space*, if each point of Y has a neighborhood whose closure is a k_ω -space.

It is implicit in a result of J. Milnor [12, Lemma 2.1] that the product of two k_ω -spaces is k_ω . This fact implies the following lemma.

LEMMA 2.7. Let Y be a locally k_ω -space. Then Y^2 is a locally k_ω -space, hence, a k -space.

LEMMA 2.8. Let $f: X \rightarrow Y$ be a closed map with each $\partial f^{-1}(y)$ Lindelöf. If X is bi - k and Y satisfies $K(\aleph_0)$, then Y is a locally k_ω -space.

PROOF. Since each closed subset of X is a bi - k , as in the proof of [7, Corollary 1.2], we can assume that each $f^{-1}(y)$ is Lindelöf. Let $y \in Y$. Then we will prove that each point of $f^{-1}(y)$ has a neighborhood contained in the inverse image of some compact subset of Y . To see this, suppose not. Then there is a point a_0 of $f^{-1}(y)$ such that for every neighborhood V of a_0 and for every compact subset K of Y , $V \not\subseteq f^{-1}(K)$.

Let $\mathcal{F} = \{X - f^{-1}(K); K \text{ is compact in } Y\}$. Then $\mathcal{F}$ is a filter base accumulating at the point a_0 . Since X is bi - k , there exists a k -sequence (A_n) in X such that $a_0 \in \overline{F \cap A_n}$ for all $n \in \omega$ and all $F \in \mathcal{F}$. Obviously, $(f(A_n))$ is a k -sequence in Y . Thus, by condition $K(\aleph_0)$, some $f(A_{n_0})$ is compact. Let $K_0 = f(A_{n_0})$. Then, $a_0 \in (X - f^{-1}(K_0)) \cap A_{n_0} \subset (X - f^{-1}(K_0)) \cap f^{-1}(K_0) = \emptyset$. This is a contradiction.

Thus, each point x of $f^{-1}(y)$ has a neighborhood V_x which is contained in the inverse image of some compact subset of Y . Since $f^{-1}(y)$ is Lindelöf, $\{V_x: x \in f^{-1}(y)\}$ contains a countable subcover $\{V_n\}_{n \in \omega}$ of $f^{-1}(y)$. For each n , let K_n be a compact subset of Y such that $V_n \subset f^{-1}(K_n)$. Since f is closed and Y is regular there exists a neighborhood W of Y such that $f^{-1}(\overline{W}) \subset \bigcup_{n \in \omega} V_n$. Let $F = f^{-1}(\overline{W})$ and $\mathcal{V} = \{F \cap V_i: i \in \omega\}$. Then, since $\mathcal{V}$ is an open covering of F , F has the weak topology with respect to $\mathcal{V}$. Since $F \cap V_i \subseteq F \cap f^{-1}(K_i)$ for each $i \in \omega$, F has the weak topology with respect to $\{F \cap f^{-1}(K_i); i \in \omega\}$. Since $f|_F$ is closed, hence quotient, $f(F) = \overline{W}$ has the weak topology with respect to $\{\overline{W} \cap K_i; i \in \omega\}$. Thus $\overline{W}$ is a k_ω -space, and so Y is a locally k_ω -space. $\square$

LEMMA 2.9. *Let $f: X \rightarrow Y$ be a closed map with X normal and $t(Y) \leq \omega$. If Y^2 is a k -space, then either Y satisfies condition $K(\aleph_0)$ or each $\partial f^{-1}(y)$ is countably compact.*

PROOF. According to [20, Theorem 4.2], if the product of two spaces is quasi- k , and one factor is not an inner-one A -space in the sense of E. Michael, R. C. Olson and F. Siwiec [11], then the other factor satisfies $K(\alpha)$, where α is its tightness. Y satisfies condition $K(\aleph_0)$, or Y is an inner-one A -space. If Y is inner-one A , by [10, Theorem 9.9] each $\partial f^{-1}(y)$ is countably compact.

LEMMA 2.10 [10]. *Bi- k -spaces are preserved by perfect images and countable products.*

By invoking Corollary 1.5, and Lemmas 2.7, 2.8, 2.9 and 2.10, we obtain the following theorem.

THEOREM 2.11 (CH). *Let $f: X \rightarrow Y$ be a closed map with X paracompact bi- k . If $t(Y) \leq \omega$, then the following are equivalent. When Y is sequential, the CH assumption can be omitted.*

- (a) Y^2 is a k -space.
- (b) Y is locally k_ω , or each $\partial f^{-1}(y)$ is compact.
- (c) Y is locally k_ω , or bi- k .

COROLLARY 2.12. *Let $f: X \rightarrow Y$ be a closed map with X or Y sequential. Let X be a paracompact space of pointwise countable type. Then Y^2 is a sequential space (equivalently, a k -space by [18, Theorem 2.2]) if and only if Y is locally k_ω or bi- k .*

Before proceeding with the next lemma, we remind the reader that the perfect image of an $\aleph$ -space is an $\aleph$ -space, but the closed image of a locally compact metric space need not be $\aleph$ -space (cf. [21, Theorem 7]).

LEMMA 2.13. *Let $f: X \rightarrow Y$ be a closed map with each $\partial f^{-1}(y)$ Lindelöf. If X is an $\aleph$ -space, and Y satisfies condition $K(\aleph_0)$, then Y is also an $\aleph$ -space.*

PROOF. Let $\mathcal{P} = \bigcup_{i=1}^{\infty} \mathcal{P}_i$ be a σ -locally finite k -network for X satisfying the following conditions: Each element of $\mathcal{P}$ is closed, $\mathcal{P}_i \subseteq \mathcal{P}_{i+1}$ and $\mathcal{P}_i$ is closed with respect to finite intersections. Let K be an arbitrary compact subset of Y . Since each subset of an $\aleph$ -space is an $\aleph$ -space, as in the proof of [7, Corollary 1.2], we can assume that each $f^{-1}(y)$ is Lindelöf and that there exists a compact subset C of X with $f(C) = K$.

Let $\mathcal{P}' = \{P \in \mathcal{P}: \mathcal{P} \cap C \neq \emptyset\}$, and let $\mathcal{C}$ be the collection of finite unions of elements of $\mathcal{P}'$ which contain the compact subset C . Then $\mathcal{C}$ is a nonempty, countable collection in X .

Let $\mathcal{C} = \{P_i: i \in \omega\}$ and $C_n = \bigcap_{i=1}^n P_i$ for each n . Then (C_n) is a k -sequence for C . Since $(f(C_n))$ is a k -sequence for K , by $K(\aleph_0)$ there exists a compact subset $f(C_{n_0})$ of Y . On the other hand, by the conditions of the collection $\mathcal{P}$, each C_n can be expressed as a union of finitely many elements of $\mathcal{P}$. So, the compact subset $f(C_{n_0})$ containing K can be expressed as a union of finitely many elements of $f(\mathcal{P})$. Let $\mathcal{K}_i = \{f(P): P \in \mathcal{P} \text{ and } f(P) \text{ is compact in } Y\}$, and let $\mathcal{K}_i^*$ be the union of all elements of $\mathcal{K}_i$. Then, since $f(\mathcal{P}_i) \subset f(\mathcal{P}_{i+1})$, by the above, each compact subset of Y is contained in some $\mathcal{K}_i^*$.

We will now prove that Y is an $\aleph$ -space. Each $\mathcal{K}_i$ is a hereditarily closure-preserving collection of compact subsets of Y , that is, whenever a subset K' of K is chosen for each $K \in \mathcal{K}_i$, the collection $\{K': K \in \mathcal{K}_i\}$ is closure-preserving. This is because $\mathcal{K}_i$ is the image of a locally finite, hence hereditarily closure-preserving, collection under a closed map. Then by a result of Michael [6, Theorem 1], each $\mathcal{K}_i^*$ is paracompact. Next, to see each $\mathcal{K}_i^*$ is locally $\aleph_0$, let $\mathcal{U}_i = \{P \in \mathcal{P}: f(P) \in \mathcal{K}_i\}$ and let $\mathcal{U}_i^* = \bigcup \mathcal{U}_i$. Then $\mathcal{U}_i^*$ has the weak topology with respect to the locally finite closed collection $\mathcal{U}_i$. Also, $f|_{\mathcal{U}_i^*}$ is closed, hence quotient. Thus $\mathcal{K}_i^* = f(\mathcal{U}_i^*)$ has the weak topology with respect to $\mathcal{K}_i$. Since f is closed and each $f^{-1}(y)$ is Lindelöf, $\mathcal{K}_i$ is locally countable. Hence each $\mathcal{K}_i^*$ is a locally k_ω -space. Since each compact subset of X is an $\aleph_0$ -space, by [8, G] each compact subset of Y is also $\aleph_0$ because it is the image of a compact subset of X . Then each $\mathcal{K}_i^*$ is a locally $\aleph_0$ -space, since each point has a neighborhood which has the weak topology with respect to a countable collection of compact $\aleph_0$ -spaces (see [8]). So, each $\mathcal{K}_i^*$ is a paracompact, locally $\aleph_0$ -space. It follows that each $\mathcal{K}_i^*$ is also an $\aleph$ -space. As is seen, each compact subset of Y is contained in some $\mathcal{K}_i^*$. Since each $\mathcal{K}_i^*$ is an $\aleph$ -space, it follows that Y is also an $\aleph$ -space. This completes the proof of the lemma.

LEMMA 2.14 [19, THEOREM 3.1]. *Let Y be a k - and $\aleph$ -space. Then Y^2 is a k - and $\aleph$ -space if and only if Y is metrizable, or Y has the weak topology with respect to a countable covering of closed and locally compact subsets of Y .*

Let a k -space Y be the closed image of an $\aleph$ -space. Since each closed subset of an $\aleph$ -space is easily seen to be a G_δ -set, each point of Y is a G_δ -set. Thus by [10, Theorem 7.3], Y is sequential. Therefore, by Corollary 1.7, and Lemmas 2.13 and 2.14, we have

THEOREM 2.15. *Let $f: X \rightarrow Y$ be a closed map with X a paracompact $\aleph$ -space. Then Y^2 is a k -space if and only if Y is metrizable, or Y is an $\aleph$ -space having the weak topology with respect to a countable covering of closed and locally compact subsets of Y .*

REMARK. Let X be an $\aleph$ -space each of whose countable (resp. uncountable) subset has an accumulation point. Then X is an $\aleph_0$ -space, and so X is compact (resp. Lindelöf). Thus, by Theorem 1.6, we have the following.

If an $\aleph$ -space X is more generally strongly collectionwise Hausdorff, then the statement of Theorem 2.15 is also valid.

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